THE MINIMAL GRADED RESOLUTION OF SOME GORENSTEIN RINGS

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Abstract. We give an explicit minimal graded resolution of a semigroup ring $K[S]$, where K is a field and $S = \langle n_1, n_2, n_3, n_4\rangle$ is a 4-generated, symmetric, non-complete intersection numerical semigroup. If $T = \langle an_1, an_2, an_3, an_4, b\rangle$, where $(a, b) = 1$, $a > 1$, and $b \in S \setminus \{n_1, n_2, n_3, n_4\}$, Watanabe showed that T is also symmetric and so $K[T]$ is a Gorenstein ring. It turns out that $K[T]$ has always Betti numbers $(1, 6, 10, 6, 1)$. We give an explicit minimal graded resolution of such semigroup rings $K[T]$ and observe that from the degrees of the Betti numbers the Hilbert series is also determined. We also show more generally how the Betti numbers of $K[T]$ are determined by the Betti numbers of $K[S]$.

1. Introduction

The positive integers $n_1, n_2, \dots, n_k$ with $\gcd(n_1, n_2, \dots, n_k) = 1$ generate the numerical semigroup $S = \langle n_1, n_2, \dots, n_k\rangle = \{\sum m_i n_i\}$, where the m_i's are non-negative integers. It is easy to see that $\mathbb{N} \setminus S$ is finite. Let $g(S)$ be the largest integer not in S (the Frobenius number of S). The semigroup S is symmetric if, for each $n \in \mathbb{Z}$, exactly one of n or $g(S) - n$ belongs to S. If K is a field and $S = \langle n_1, n_2, \dots, n_k\rangle$, then $K[S] = K[t^{n_1}, \dots, t^{n_k}]$ is the semigroup ring of S. It is shown by Kunz [4] that $K[S]$ is a Gorenstein ring if and only if S is symmetric. Let $\phi\colon K[x_1, \dots, x_k] \to K[t]$ be defined by $x_i \mapsto t^{n_i}$. If we let $\deg x_i = n_i$, this map is of degree 0 and $K[S] = K[x_1, \dots, x_k]/\ker(\phi)$.

In [8] Watanabe describes a construction giving from a semigroup S generated by k elements, a semigroup T generated by $k+1$ elements. Namely if $S = \langle n_1, \dots, n_k\rangle$, then $T = \langle an_1, \dots, an_k, b\rangle$, where $(a, b) = 1$, $a > 1$, and $b \in S \setminus \{n_1, \dots, n_k\}$. It is shown in [8] that $K[T]$ is Gorenstein (a complete intersection) if and only if $K[S]$ is Gorenstein (a complete intersection). Furthermore, it is shown in [8] that the relations $\ker(K[x_1, \dots, x_{k+1}] \to K[t])$ of $K[T]$ are the "same" as the relations of $K[S]$ plus one extra.

If $R = K[x_1, \dots, x_n]/I$ is any graded K-algebra and $A = K[x_1, \dots, x_n]$, then R has a minimal graded A-resolution

$$\mathbf{F}_* : 0 \longrightarrow A^{\beta_m} \longrightarrow \cdots \longrightarrow A^{\beta_1} \xrightarrow{\phi_1} A \longrightarrow 0$$

where $\mathrm{Im}\phi_1 = I$. The β_i's are called the Betti numbers of R. If we let all maps be of degree 0, then $A^{\beta_i} = \oplus_j A[-j]^{\beta_{i,j}}$, where $A[-j]_d = A_{d-j}$, and the $\beta_{i,j}$'s are called the graded Betti numbers.

In [1] Bresinsky determines $\ker(\phi)$ for S symmetric and generated by four elements. We will determine a minimal resolution of $K[S]$ in this case. Then we will also determine a minimal resolution of $K[T]$, where T comes from S via Watanabe's construction.

We will use the following well known facts:

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1. If S is generated by k elements, then the resolution has length $\operatorname{codim} K[S] = k-1$ since $K[S]$ is a 1-dimensional Cohen-Macaulay ring.

2. The alternating sum of the Betti numbers is zero.

3. It is usual to define the Betti numbers of $R = K[x_1, \ldots, x_n]/I$ as

$$\beta_i = \dim_K H_i(\mathbf{F}_* \otimes K) = \dim_K \operatorname{Tor}_i^A(R, K).$$

This gives us an alternative way to define the Betti numbers, since also $\operatorname{Tor}_i^A(R, K) = H_i(\mathbf{G}_* \otimes R)$, where $\mathbf{G}_*$ is a minimal A-resolution of K (the Koszul complex).

4. If R is Cohen-Macaulay, the highest nonzero Betti number is called the CM-type of R.

5. If we concentrate $\mathbf{F}_*$ above to a certain degree d, we get an exact sequence of vector spaces

$$0 \longrightarrow \oplus_j (A[-j]^{\beta_{m,j}})_d \longrightarrow \cdots \longrightarrow \oplus_j (A[-j]^{\beta_{1,j}})_d \longrightarrow A_d \longrightarrow (A/I)_d \longrightarrow 0$$

so

$$0 \longrightarrow \oplus_j A_{d-j}^{\beta_{m,j}} \longrightarrow \cdots \longrightarrow \oplus_j A_{d-j}^{\beta_{1,j}} \longrightarrow A_d \longrightarrow (A/I)_d \longrightarrow 0.$$

The alternating sum of the dimensions of these vector spaces is 0. Multiplying each dimension with z^d and summing for $d \geq 0$, we get

$$\operatorname{Hilb}_{A/I}(z) = \operatorname{Hilb}_A(z)(1 + \sum_{i=1}^{m} \sum_j (-1)^i \beta_{i,j} z^j).$$

If $\deg(x_i) = n_i$, then $\operatorname{Hilb}_{K[x_1,\ldots,x_n]}(z) = 1/\prod_{i=1}^{n}(1 - z^{n_i})$.

2. Betti numbers

Let $S = \langle n_1, \ldots, n_k \rangle$ and $T = \langle an_1, \ldots, an_k, b \rangle$, where $(a, b) = 1$, $a > 1$, and $b \in S \setminus \{n_1, \ldots, n_k\}$.

Theorem 1. $\beta_i(K[T]) = \beta_i(K[S]) + \beta_{i-1}(K[S])$*, for* $i = 1, \ldots, k$*.*

Proof. If $K[S] = k[x_1, \ldots, x_k]/I$, then $K[T] = k[x_1, \ldots, x_k, y]/(I, f)$, where $f = y^a - \sum_{i=1}^{m} x_i^{c_i}$, see [8]. Let $A = K[x_1, \ldots, x_k]$ and $B = A[y]$. Since f is a nonzero-divisor of $K[S][y]$, we have the exact sequence of B-modules:

$$0 \longrightarrow K[S][y] \xrightarrow{\cdot f} K[S][y] \longrightarrow K[T] \longrightarrow 0.$$

Since $\operatorname{Tor}_i^B(K[S][y], K) = \operatorname{Tor}_i^A(K[S], K)$, we get a long exact sequence of K-vector spaces

$$0 \longrightarrow \operatorname{Tor}_k^B(K[T], K) \longrightarrow \operatorname{Tor}_{k-1}^A(K[S], K) \xrightarrow{0} \cdots$$
$$\cdots \xrightarrow{0} \operatorname{Tor}_i^A(K[S], K) \longrightarrow \operatorname{Tor}_i^B(K[T], K) \longrightarrow \operatorname{Tor}_{i-1}^A(K[S], K) \xrightarrow{0} \cdots$$
$$\cdots \xrightarrow{0} \operatorname{Tor}_1^A(K[S], K) \longrightarrow \operatorname{Tor}_1^B(K[T], K) \longrightarrow K \xrightarrow{0} K \longrightarrow K \longrightarrow 0$$

which gives the claim. □

In [8] it is shown that the CM-type of $K[T]$ is one if and only if the CM-type of $K[S]$ is one. The following generalizes this.

Corollary 2. *The CM-type of* $K[T]$ *equals the CM-type of* $K[S]$*.*

Proof. By points 1 and 4 of the Introduction, the CM-type of $K[T]$ is $\beta_k(K[T])$ and, by Theorem 1, $\beta_k(K[T]) = \beta_k(K[S]) + \beta_{k-1}(K[S]) = \beta_{k-1}(K[S])$, which is the CM-type of $K[S]$. □

Corollary 3. *If* $S = \langle n_1, n_2, n_3 \rangle$ *is not symmetric, and* $T = \langle an_1, an_2, an_3, b \rangle$*,then the Betti numbers of* $K[T]$ *are 1,4,5,2.*

Proof. We have $\beta_1(K[S]) = 3$, see [3], so $\beta_2(K[S]) = 2$, see points 1 and 2 in the introduction.Thus the Betti numbers of $K[S]$ are 1,3,2. Then by Theorem 1 the Betti numbers of $K[T]$ are 1,4,5,2. $\square$

3. The resolution of $K[S]$ when S is 4-generated symmetric

If S is generated by at most 3 elements and is symmetric, then $K[S]$ is a complete intersection, see [3]. If S is a complete intersection, the resolution is the Koszul complex, so we concentrate on the case when S is symmetric, but $K[S]$ is not a complete intersection. Thus, the first interesting case is when S is generated by 4 elements. The result in [1] is, in case S is not a complete intersection, that, for some numbering of the variables,

$$\ker(\phi) = (f_1, f_2, f_3, f_4, f_5)$$

where $f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}, f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}, f_5 = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}}$, and where $\alpha_1 = \alpha_{21} + \alpha_{31}, \alpha_2 = \alpha_{32} + \alpha_{42}, \alpha_3 = \alpha_{13} + \alpha_{43}, \alpha_4 = \alpha_{14} + \alpha_{24}$.

Let $A = K[x_1, x_2, x_3, x_4]$. We now give the whole minimal A-resolution of $K[S]$ in case S is 4-generated symmetric but not a complete intersection.

Theorem 4. *In case S is 4-generated symmetric, not a complete intersection, then the following is a minimal resolution of $K[S]$:*

$$0 \longrightarrow A \xrightarrow{\phi_3} A^5 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where $\phi_1 = (f_1, f_2, f_3, f_4, f_5)$

$$\phi_2 = \begin{pmatrix} x_2^{\alpha_{32}} & x_3^{\alpha_{43}} & x_4^{\alpha_{24}} & 0 & 0 \\ 0 & 0 & x_1^{\alpha_{31}} & x_4^{\alpha_{14}} & x_3^{\alpha_{43}} \\ x_1^{\alpha_{21}} & x_4^{\alpha_{14}} & x_2^{\alpha_{42}} & 0 & 0 \\ 0 & 0 & x_3^{\alpha_{13}} & x_1^{\alpha_{21}} & x_2^{\alpha_{32}} \\ -x_3^{\alpha_{13}} & -x_1^{\alpha_{31}} & 0 & x_2^{\alpha_{42}} & x_4^{\alpha_{24}} \end{pmatrix}$$

and $\phi_3 = (-f_4, -f_2, -f_5, f_3, f_1)^t$.

Proof. We will use [2, Theorem 20.9]. We have to show we have a complex, that the rank (the largest nonzero minor) of ϕ_1 and ϕ_3 is 1, and that $\mathrm{rank}(\phi_2) = 4$. Furthermore that the depth of $I(\phi_i)$ is at least i for all i, where $I(\phi_i)$ is the ideal of A generated by the minors of ϕ_i of size $\mathrm{rank}(\phi_i)$. A simple calculation shows that $\phi_1\phi_2 = \phi_2\phi_3 = 0$. That $\mathrm{rank}(\phi_1) = \mathrm{rank}(\phi_3) = 1$ is clear. If we delete row 1 and column 5 in ϕ_2 we get a matrix with determinant f_1^2. If we delete row 2 and column 2, we get the determinant f_2^2. These two determinants are relatively prime, so they constitute a regular sequence. In fact, $I(\phi_2) = (f_1, f_2, f_3, f_4, f_5)^2$. $\square$

Example 5. Let $S = \langle n_1, n_2, n_3, n_4 \rangle = \langle 7, 9, 8, 13 \rangle$. Then the following is a minimal resolution of $K[S]$:

$$0 \longrightarrow A \xrightarrow{\phi_3} A^5 \xrightarrow{\phi_2} A^5 \xrightarrow{\phi_1} A \longrightarrow 0$$

where $\phi_1 = (x_1^3 - x_3x_4, x_2^3 - x_1^2x_4, x_3^2 - x_1x_2, x_4^2 - x_2^2x_3, x_3x_1^2 - x_2x_4)$

$$\phi_2 = \begin{pmatrix} x_2 & x_3 & x_4 & 0 & 0 \\ 0 & 0 & x_1 & x_4 & x_3 \\ x_1^2 & x_4 & x_2^2 & 0 & 0 \\ 0 & 0 & x_3 & x_1^2 & x_2 \\ -x_3 & -x_1 & 0 & x_2^2 & x_4 \end{pmatrix}$$

and $\phi_3 = (-(x_4^2 - x_2^2x_3), -(x_2^3 - x_1^2x_4), -(x_3x_1^2 - x_2x_4), x_3^2 - x_1x_2, x_1^3 - x_3x_4)^t$.

Recall that the set of pseudo-Frobenius numbers of a numerical semigroup S is $PF(S) = \{t \in \mathbb{Z} \setminus S; t + s \in S \text{ for all } s \in S \setminus \{0\}\}$.

Lemma 6. *Let $S = \langle n_1, \ldots, n_k\rangle$, $0 \neq s \in S$, and $K[S] = K[t^{n_1}, \ldots, t^{n_k}]$. Then $n \in PF(S)$ if and only if $\overline{t^{n+s}} \in \mathrm{Soc}(K[S]/(t^s)$.*

Proof. Let $M = (t^{n_1}, \ldots, t^{n_k})$. We have $n \in PF(S)$ if and only if $t^n \notin K[S]$ and $t^nM \subseteq K[S]$, so if and only if $t^{n+s} \notin t^sK[S]$ and $t^{n+s}K[S] \subseteq t^sK[S]$, so if and only if $\overline{t^{n+s}} \neq \overline{0}$ and $\overline{t^{n+s}M} = \overline{0}$ in $K[S]/(t^s)$, so if and only if $\overline{t^{n+s}} \in \mathrm{Soc}(K[S]/(t^s))$. □

Proposition 7. *Let $S = \langle n_1, \ldots, n_k\rangle$ and let $\beta_{i,j}$ be the graded Betti numbers of $K[S]$. Then $n \in PF(S)$ if and only if $\beta_{k-1,n+N} \neq 0$ (in fact $\beta_{k-1,n+N} = 1$), where $N = \sum_{i=1}^k n_i$. In particular, if S is symmetric, then $n = g(S)$ if and only if $\beta_{k-1} = \beta_{k-1,g(S)+N}$.*

Proof. Let $s = n_1$. Then the dimension of

$$H_{k-1}(K[t^{n_1}, \ldots, t^{n_k}]/(t^{n_1})) = \mathrm{Soc}(K[t^{n_1}, \ldots, t^{n_k}]/(t^{n_1}))$$

is the highest Betti number β_{k-1} of $K[S]$, which exists in degrees $n_2 + \cdots + n_k + \deg \mathrm{Soc}(K[t^{n_1}, \ldots, t^{n_k}]/(t^{n_1}))$. Thus, by Lemma 6, $n \in PF(S)$ if and only if $\beta_{k-1,n+N} \neq 0$ (in fact $\beta_{k-1,n+N} = 1$). □

Example 8. In the proof of Proposition 7, applied to our example $R = K[S]$, where $S = \langle 7, 9, 8, 13\rangle$, the dimension of $\mathrm{Soc}(R/(t^7)) = \mathrm{Soc}(\bar{R})$ is one (in fact S is symmetric, i.e., R is Gorenstein) and $H_3(\mathbf{G}_* \otimes R) = \mathrm{Soc}(\bar{R})$ is a one dimensional vector space generated by $\overline{t^{g(S)+n_1}} = \overline{t^{26}}$. Since $\mathbf{G}_*$ is the Koszul complex of length $k-1 = 3$ in the three variables x_2, x_3, x_4 of degrees n_2, n_3, n_4, this vector space is nonzero only in degree $(g(S) + n_1) + (n_2 + n_3 + n_4) = (19 + 7) + (9 + 8 + 13) = 56$.

Corollary 9. *If $S = \langle n_1, \ldots, n_4\rangle$ is 4-generated symmetric, not a complete intersection and $N = \sum_{i=1}^4 n_i$, then $g(S) = \alpha_1n_1 + \alpha_{32}n_2 + \alpha_4n_4 - N = \alpha_2n_2 + \alpha_{14}n_4 + \alpha_3n_3 - N = \alpha_4n_4 + \alpha_{21}n_1 + \alpha_3n_3 - N = \alpha_1n_1 + \alpha_{43}n_3 + \alpha_2n_2 - N$.*

Proof. We get $\beta_3 = \beta_{3,n_1\alpha_1+n_2\alpha_{32}+n_4\alpha_4} = \beta_{3,\alpha_2n_2+\alpha_{14}n_4+\alpha_3n_3} = \beta_{3,\alpha_4n_4+\alpha_{21}n_1+\alpha_3n_3} = \beta_{3,\alpha_1n_1+\alpha_{43}n_3+\alpha_2n_2}$ by adding the degrees in the resolution given in Theorem 4. Then we use Proposition 7. □

Corollary 10. *We always have $\alpha_1n_1 + \alpha_{32}n_2 + \alpha_4n_4 = \alpha_2n_2 + \alpha_{14}n_4 + \alpha_3n_3 = \alpha_4n_4 + \alpha_{21}n_1 + \alpha_3n_3 = \alpha_1n_1 + \alpha_{43}n_3 + \alpha_2n_2$.*

4. The resolution of some 5-generated semigroup rings

In case $S = \langle n_1, n_2, n_3, n_4\rangle$ is symmetric 4-generated and not a complete intersection, we have seen that the Betti numbers are $(\beta_0, \beta_1, \beta_2, \beta_3) = (1, 5, 5, 1)$. All 5-generated symmetric and not a complete intersection semigroups do not have the same Betti numbers. If $S = \langle 6, 7, 8, 9, 10\rangle$, the Betti numbers are $(1, 9, 16, 9, 1)$. We will consider the case of semigroups which come from 4-generated semigroups as above via Watanabe's construction. By Theorem 1 they all have Betti numbers $(1, 6, 10, 6, 1)$. We will now determine the whole resolution. Let $S = \langle n_1, n_2, n_3, n_4\rangle$

be symmetric and not a complete intersection, and $T = \langle an_1, an_2, an_3, an_4, b\rangle$, where $(a, b) = 1$, $a > 1$, and $b \in S \setminus \{n_1, n_2, n_3, n_4\}$, $b = \sum_{i=1}^{4} c_i n_i$. Keeping the notation of previous section for the relations f_i, $1 \le i \le 5$ in the resolution of $K[S]$ and setting $A = K[x_1, x_2, x_3, x_4, x_5]$, we have

Theorem 11. *In case S is 4-generated symmetric, not a complete intersection, then the following is a minimal resolution of $K[T]$:*

$$0 \longrightarrow A \xrightarrow{\Phi_4} A^6 \xrightarrow{\Phi_3} A^{10} \xrightarrow{\Phi_2} A^6 \xrightarrow{\Phi_1} A \longrightarrow 0$$

where $\Phi_1 = (f_1, f_2, f_3, f_4, f_5, f_6)$, with $f_6 = x_5^a - \sum_{i=1}^{4} x_i^{c_i}$,

$$\Phi_2 = \begin{pmatrix} x_2^{\alpha_{32}} & x_3^{\alpha_{43}} & x_4^{\alpha_{24}} & 0 & 0 & -f_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1^{\alpha_{31}} & x_4^{\alpha_{14}} & x_3^{\alpha_{43}} & 0 & -f_6 & 0 & 0 & 0 \\ x_1^{\alpha_{21}} & x_4^{\alpha_{14}} & x_2^{\alpha_{42}} & 0 & 0 & 0 & 0 & -f_6 & 0 & 0 \\ 0 & 0 & x_3^{\alpha_{13}} & x_1^{\alpha_{21}} & x_2^{\alpha_{32}} & 0 & 0 & 0 & -f_6 & 0 \\ -x_3^{\alpha_{13}} & -x_1^{\alpha_{31}} & 0 & x_2^{\alpha_{42}} & x_4^{\alpha_{24}} & 0 & 0 & 0 & 0 & -f_6 \\ 0 & 0 & 0 & 0 & 0 & f_1 & f_2 & f_3 & f_4 & f_5 \end{pmatrix},$$

$$\Phi_3 = \begin{pmatrix} f_6 & 0 & 0 & 0 & 0 & -f_4 \\ 0 & f_6 & 0 & 0 & 0 & -f_2 \\ 0 & 0 & f_6 & 0 & 0 & -f_5 \\ 0 & 0 & 0 & f_6 & 0 & f_3 \\ 0 & 0 & 0 & 0 & f_6 & f_1 \\ x_2^{\alpha_{32}} & x_3^{\alpha_{43}} & x_4^{\alpha_{24}} & 0 & 0 & 0 \\ 0 & 0 & x_1^{\alpha_{31}} & x_4^{\alpha_{14}} & x_3^{\alpha_{43}} & 0 \\ x_1^{\alpha_{21}} & x_4^{\alpha_{14}} & x_2^{\alpha_{42}} & 0 & 0 & 0 \\ 0 & 0 & x_3^{\alpha_{13}} & x_1^{\alpha_{21}} & x_2^{\alpha_{32}} & 0 \\ -x_3^{\alpha_{13}} & -x_1^{\alpha_{31}} & 0 & x_2^{\alpha_{42}} & x_4^{\alpha_{24}} & 0 \end{pmatrix},$$

and $\Phi_4 = (f_4, f_2, f_5, -f_3, -f_1, f_6)^t$.

Proof. We use the same method as in Theorem 4. We have to show we have a complex, that $\operatorname{rank}(\Phi_1) = \operatorname{rank}(\Phi_4) = 1$, $\operatorname{rank}(\Phi_2) = \operatorname{rank}(\Phi_3) = 4$, and that the depth of $I(\Phi_i)$ is at least i for all i. It is easy to check that $\Phi_3\Phi_4 = \Phi_2\Phi_3 = \Phi_1\Phi_2 = 0$. In particular, to show that $\Phi_3\Phi_4 = 0$, we use the fact that for the maps of Theorem 4, we have $\phi_2\phi_3 = 0$. If we delete row 1 and columns 5,6,7,9,10 in Φ_2 we get a matrix with determinant f_1^3, so $\operatorname{rank}(\Phi_2) = 4$. Similarly $\operatorname{rank}(\Phi_3) = 4$. If we delete row 2 and columns 2,6,7,8,9 in Φ_2, we get the determinant f_2^3. These determinants are relatively prime. If we delete row 6 and columns 1,2,3,4,5 we get

the determinant $-f_6^5$, which obviously is a nonzerodivisor modulo the two first. Thus the depth of $I(\Phi_2)$ is at least 3. Similarly depth$(I(\Phi_3)) \geq 3$. In fact, also the dualized complex is a resolution, so $I(\Phi_2) = I(\Phi_3)$. □

Corollary 12. *Let $T = \langle an_1, an_2, an_3, an_4, b\rangle$, where $S = \langle n_1, n_2, n_3, n_4\rangle$ is symmetric but not a complete intersection, $a > 1$ and $b \in S \setminus \{n_1, \dots, n_k\}$, $(a, b) = 1$. Suppose that the degrees of the first Betti numbers of $k[S]$ are u_1, u_2, u_3, u_4, u_5 and that the degrees of the second Betti numbers are v_1, v_2, v_3, v_4, v_5. Then the degrees of the first Betti numbers of $K[T]$ are $au_1, au_2, au_3, au_4, au_5, ab$, the degrees of the second Betti numbers are $av_i, 1 \leq i \leq 5, ab + au_i, 1 \leq i \leq 5$, the degrees of the third Betti numbers are $ab + av_i, 1 \leq i \leq 5, au_4 + av_1$, the degree of the fourth Betti number is $au_4 + ab + av_1$.*

Proof. Note first that in the minimal resolution of $K[T]$ given in Theorem 11 f_i are relations of degrees au_i , for $1 \leq i \leq 5$ and f_6 is a relation of degree ab. We can read the Betti numbers from the resolution of $K[T]$. The first Betti numbers are in fact the degrees of the relations f_i, for $1 \leq i \leq 6$. Since the columns of matrix ϕ_2 in Theorem 4 are by hypothesis homogeneous elements of $A^5 = \oplus_{i=1}^5 A[-u_i]$ of degrees $v_1, \dots, v_5$, then the columns of matrix Φ_2 in Theorem 11 are homogeneous elements of $A^6 = \oplus_{i=1}^5 A[-au_i] \oplus A[-ab]$ of degrees $av_1, \dots, av_5, ab + au_1, \dots, ab + au_5$. Now the six columns of matrix Φ_3 in Theorem 11 are homogeneous elements of $A^{10} = \oplus_{i=1}^5 A[-av_i] \oplus_{i=1}^5 A[-(ab + au_i)]$ of degrees $ab + av_1, \dots, ab + av_5, au_4 + av_1$. Finally the unique column of matrix Φ_4 in Theorem 11 is a homogeneous element of $A^6 = \oplus_{i=1}^5 A[-(ab + av_i)] \oplus A[-(au_4 + av_1)]$ of degree $au_4 + ab + av_1$. □

Remark 13. For simplicity we used in Corollary 12 new symbols, but we could also express the degrees of Betti numbers of $K[T]$ in terms of n_i, α_j, α_{ml}. For example the degree of the fourth Betti number is $au_4 + ab + av_1 = a\alpha_4 n_4 + ab + a(\alpha_1 n_1 + \alpha_{32} n_2)$.

Corollary 14. *Let $T = \langle an_1, an_2, an_3, an_4, b\rangle$, where $S = \langle n_1, n_2, n_3, n_4\rangle$ is symmetric but not a complete intersection, $a > 1$ and $b \in S \setminus \{n_1, \dots, n_k\}$, $(a, b) = 1$. Then the Frobenius number of T is*

$$g(T) = ag(S) + b(a - 1).$$

Proof. By Proposition 7 the degree of the highest Betti number of $K[S]$ (the third) is $g(S) + n_1 + n_2 + n_3 + n_4$. Looking at matrix ϕ_3 of Theorem 4, we see that it is also equal to $u_4 + v_1$. Hence $g(S) = u_4 + v_1 - (n_1 + n_2 + n_3 + n_4)$. By Proposition 7 the degree of the highest Betti number of $K[T]$ (the fourth) is $g(T) + an_1 + an_2 + an_3 + an_4 + b$ and by Corollary 12 it is also equal to $au_4 + ab + av_1$. Hence $g(T) = a(g(S) + b) - b = ag(S) + b(a - 1)$.

□

Remark 15. If $S = \langle n_1, \dots, n_k\rangle$, with pseudo-Frobenius numbers $PF(S) = \{g_1, \dots, g_h\}$, Garcia-Sanchez and Rosales in [6] (see also [7]) consider the semigroup $T = \langle 2n_1, \dots, 2n_k, b - 2g_1, \dots, b - 2g_h\rangle$, where b is an odd number $\geq 3g(S) + 1$.They show that $2s \in T$ if and only if $s \in S$, that is $T/2 = S$. As we see, there are infinitely many such T. In case S is symmetric, $PF(S) = \{g(S)\}$, so the only new generator is an odd number $b \geq g(S) + 1$, and in that case their construction is a special case of Watanabe's one with $a = 2$. It is shown in [7] that in this case the Frobenius number of T is $g(T) = 2g(S) + b$.

Corollary 14 is a generalization, in our particular situation, of that result.

We also observe that it is not necessary to assume that the last odd generator b of the construction of Garcia-Sanchez and Rosales is $\geq g(S) + 1$, it suffices that it belongs to $S \setminus \{n_1, n_2, n_3, n_4\}$. Let e be the smallest nonzero element in S (the

multiplicity), and let $w_1 < w_2 < \cdots < w_e$ be the smallest elements in each residue class mod e (the Apery set mod e). Thus $w_1 = 0$ and $w_e = g(S) + e$. If S is symmetric then $w_i + w_{e-i} = w_e$ (cf. e.g. [7, Proposition 4.10]). Then, if $e \neq 2$, all generators are smaller than $g(S)$. Hence the hypothesis $b \geq g(S) + 1$ is stronger than the hypothesis $b \in S \setminus \{n_1, n_2, n_3, n_4\}$. If e.g. $S = \langle 7, 8, 9, 13 \rangle$, then $g(S) = 19$, but also $b = 15$ and $b = 17$ are good, so $\langle 14, 16, 18, 26, 15 \rangle$ and $\langle 14, 16, 18, 26, 17 \rangle$ satisfy the claim and give Gorenstein rings.

Example 16. If $S = \langle 7, 8, 9, 13 \rangle$, then $\beta_{1,i}(S) = 1$ for $i = 16, 21, 22, 26, 27$, $\beta_{2,i}(S) = 1$ for $i = 29, 30, 34, 35, 40$, and $\beta_{3,i}(S) = 1$ for $i = 56$. So, if $T = \langle 14, 16, 18, 26, 15 \rangle$, then $\beta_{1,i}(T) = 1$ for $i = 32, 42, 44, 52, 54, 30$, $\beta_{2,i}(T) = 1$ for $i = 58, 60, 68, 70, 80, 62, 72, 74, 82, 84$. It is easy to see that $g(S) = 19$, thus $\beta_{4,i}(T) = 1$ for $i = 14 + 16 + 18 + 26 + 30 + 38 = 142$. Finally, $\beta_{3,i}(T) = 1$ for $i = 112, 110, 100, 98, 90, 88$.

Corollary 17. *If we in Corollary 12 replace b by $b + a$, the degrees of the first Betti numbers of $K[T]$ increase by $(0, 0, 0, 0, 0, a^2)$, the degrees of the second by $(0, 0, 0, 0, 0, a^2, a^2, a^2, a^2, a^2)$, of the third by $(0, a^2, a^2, a^2, a^2, a^2)$, and of the fourth by a^2.*

Example 18. If $T = \langle 14, 16, 18, 26, 17 \rangle$, then $\beta_{1,i}(T) = 1$ for $i = 32, 42, 44, 52, 54, 34$, $\beta_{2,i}(T) = 1$ for $i = 58, 60, 68, 70, 80, 66, 76, 78, 86, 88$, $\beta_{3,i} = 1$ for $i = 112, 114, 104, 102, 94, 92$, and $\beta_{4,i} = 1$ for $i = 146$.

Example 19. Using point 5 of the introduction, we can also determine the Hilbert series. Let $S = \langle 5, 6, 7, 8 \rangle$ and $T = \langle 10, 12, 14, 16, 11 \rangle$. Then $K[S]$ has Betti numbers $\beta_{1,j} = 1$ for $j = 12, 13, 14, 15, 16$, $\beta_{2,j} = 1$ for $j = 19, 20, 21, 22, 23$ and $\beta_{3,35} = 1$, so $K[T]$ has Betti numbers $\beta_{1,j} = 1$ for $j = 22, 24, 26, 28, 30, 32$, $\beta_{2,j} = 1$ for $j = 38, 40, 42, 44, 48, 50, 52, 54$, $\beta_{2,46} = 2$, $\beta_{3,j} = 1$ for $j = 60, 62, 64, 66, 68, 70$, $\beta_{4,92} = 1$. The Hilbert series of $K[T]$ is
$1 + \sum_{i=1}^{4}((-1)^i \beta_{i,j} z^j)/((1 - z^{10})(1 - z^{12})(1 - z^{14})(1 - z^{16})(1 - z^{11})) = (1 - z^{22} - z^{24} - z^{26} - z^{28} - z^{30} - z^{32} + z^{38} + z^{40} + z^{42} + z^{44} + 2z^{46} + z^{48} + z^{50} + z^{52} + z^{54} - z^{60} - z^{62} - z^{64} - z^{66} - z^{68} - z^{70} + z^{92})/((1 - z^{10})(1 - z^{12})(1 - z^{14})(1 - z^{16})(1 - z^{11})) = 1 + z^{10} + z^{11} + z^{12} + z^{14} + z^{16} + z^{20} + z^{21} + z^{22} + z^{23} + z^{24} + z^{25} + z^{26} + z^{27} + z^{28} + z^{30}/(1 - z)$.

Remark 20. In case of the construction of Garcia-Sanchez and Rosales, the total Betti numbers of $K[T]$, $(1, 6, 10, 6, 1)$, were determined by Micale and Olteanu [5].

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